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ITERATIVE METHODS OF INVESTIGATING PARAMETRICALLY EXCITED LINEAR DYNAMIC SYSTEMS*

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Iterative methods of determining the characteristic exponents (both isolated values and the full set of exponents) for a system of linear differential equations with periodic coefficients whose matrix is nearly constant are considered. The algorithm for reduction to a linear system with a constant matrix is based on the Newton-Kantorovich method of solving non-linear matrix equations. The iterative algorithm is generalized to the case with close characteristic exponents.

The method is used for a numerical analysis of the dynamics of a magnetically suspended vehicle which moves with a constant velocity along a guideway represented by elastic dynamically uncoupled homogeneous beams /1, 2/. A covariance analysis is carried out of the random process of the excitation of vibrations in the system, allowing for the unevenness of the guideway. The characteristic exponents and multipliers for the system are calculated.

1. Statement of the problem. Consider a dynamic system with parametric excitation described by the linear equation

$$x^{\cdot} = (A_0 + A_1(t))x + f(t) \tag{1.1}$$

with a constant $(m \times m)$ matrix A_0, a 2π-periodic matrix of parametric excitations A_1, the state vector x and the perturbation vector $f(t)$. The norm of the matrix A_1 is small; initially we assume that A_0 has a simple structure. We introduce the matrix R_0 of the right eigenvectors of the matrix A_0 and make the change of variables $y = R_0 x$. In the new variables, Eq.(1.1) takes the form

$$y^{\cdot} = (\Lambda_0 + B(t))y + g(t),\ \Lambda_0 = \operatorname{diag}\{\lambda_{0j}\} \tag{1.2}$$

where λ_{0j} are the eigenvalues of the matrix A_0. Without loss of generality, we will assume that the Fourier series of $B(t)$

$$B(t) = \sum_k B_k e^{ikt}$$

does not contain B_0.

For a sufficiently small matrix $B(t)$, each isolated eigenvalue λ_{0j} generates a finite number of characteristic exponents of the equation with periodic coefficients (1.2) $\lambda_j + ik$ ($\lambda_j \approx \lambda_{0j}$; $k = 0, \pm 1, \pm 2, \ldots$), and also the multiplier $\rho_j = \exp(2\pi\lambda_j)$ with the corresponding solution of Eq.(1.2) $y_j(t)$, $y_j(t+2\pi) = \rho_j y_j(t)$. The solution $y_j(t)$ may be expressed in terms of the characteristic exponents and the constant column vectors which are analogoues of the eigenvectors of the system with constant coefficients.

$$y_j(t) = \Sigma r_{jk} \exp((\lambda_j + ik)t)$$

A complete analysis of a system with periodic coefficients requires a computation of the characteristic exponents and the corresponding vectors r_{jk}.

2. Iterative computation of an isolated characteristic exponent. Consider the characteristic exponent λ generated by the isolated eigenvalue λ_0 and the corresponding collection of vectors r_k. For $B(t) = 0$ we have $\lambda = \lambda_0$, $r_k = 0$, $k = \pm 1, \pm 2, \ldots$; $r_0 = e_1 = (1, 0, \ldots, 0)^T$. Without loss of generality, we assume that $r_0 = (1, \Delta r^T)^T$. Substituting the expression $y(t) = \Sigma r_k e^{(\lambda+ik)t}$ into (1.2), we obtain a system of matrix equations for λ and r_k:

$$((\lambda + ik)E - \Lambda_0) r_k = \sum_l B_l r_{k-l},\quad k = 0, \pm 1, \pm 2, \ldots \tag{2.1}$$

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Let

$$B_k = (b_{1k}, B_{1k}^T)^T = (b_{2k}, B_{2k}), \quad \Delta\lambda = \lambda - \lambda_0, \quad \Lambda_0 = \operatorname{diag}\{\lambda_0, \Lambda_{01}\},$$

Here b_{1k}^T and b_{2k} are respectively the first row and the first column of the matrix B_k. Consider the equation for $k = 0$ in system (2.1). In our notation, it separates into two equations:

$$\Delta\lambda = \Sigma b_{1l}{}^T r_{-l}, \ \Delta r = U_0(-\Delta\lambda\Delta r + \Sigma B_{1l} r_{-l}); \ U_0 = (\lambda_0 E - \Lambda_{01})^{-1}$$

(E is the identity matrix). For the remaining equations in (2.1), we must stipulate that the characteristic exponents generated by λ_0 are isolated. If the matrix $B(t)$ is small, a sufficient condition for this is $\lambda_0 - \lambda_{0j} \neq ik, \ k = 0, \pm 1, \pm 2, \ldots$. Then the matrices $U_k = ((\lambda_0 + ik)E - \Lambda_0)^{-1}$ exist and we have

$$r_k = U_k(-\Delta\lambda r_k + b_{2k} + B_{2k}\Delta r + \Sigma B_l r_{k-l})$$

Now let us consider the iteration process

$$\begin{gathered}\Delta\lambda^{(n)} = \Sigma b_{1l}{}^T r_{-l}^{(n-1)}, \quad \Delta r^{(n)} = U_0(-\Delta\lambda^{(n)}\Delta r^{(n-1)} + \Sigma B_{1l} r_{-l}^{(n-1)}) \\ r_k^{(n)} = U_k(-\Delta\lambda^{(n)} r_k^{(n-1)} + b_{2k} + B_{2k}\Delta r^{(n-1)} + \Sigma B_l r_{k-l}^{(n-1)}) \\ n = 1, 2, \ldots, \ r_k^{(0)} = 0, \ \Delta r^{(0)} = 0, \ \Delta\lambda^{(0)} = 0\end{gathered} \tag{2.2}$$

and investigate its convergence. Put

$$\delta r_0^{(n)} = \Delta r^{(n)} - \Delta r^{(n-1)}, \quad \delta r_k^{(n)} = r_k^{(n)} - r_k^{(n-1)}, \quad \delta\lambda^{(n)} = \Delta\lambda^{(n)} - \Delta\lambda^{(n-1)}$$

Using this notation, we obtain from (2.2)

$$\begin{gathered}\delta r_0^{(n)} = U_0(-\delta\lambda^{(n)}\Delta r^{(n-1)} - \Delta\lambda^{(n-1)}\delta r_0^{(n-1)} + \Sigma B_{1l}\delta r_{-l}^{(n-1)}) \\ \delta r_k^{(n)} = U_k(-\delta\lambda^{(n)} r_k^{(n-1)} - \Delta\lambda^{(n-1)}\delta r_k^{(n-1)} + B_{2k}\delta r_0^{(n-1)} + \Sigma B_l \delta r_{k-l}^{(n-1)}) \\ \delta\lambda^{(n)} = \Sigma b_{1l}{}^T \delta r_{-l}^{(n-1)}, \quad n = 1, 2, \ldots, \quad \delta r_k^{(1)} = U_k b_{2k}, \ \delta r_0^{(1)} = 0\end{gathered}$$

Introducing compatible norms and using the notation

$$\begin{gathered}B = \Sigma \| B_j \|, \ b_1 = \Sigma \| b_{1j} \|, \ b_2 = \max \| b_{2k} \|, \ \Delta = \min |\lambda_0 + ik - \lambda_{0j}| \\ c^{(n)} = \max \{\| r_k^{(n)} \|, \| \Delta r^{(n)} \|\}, \ \delta c^{(n)} = \max \| \delta r_k^{(n)} \|, \ k = 0, \pm 1, \pm 2, \ldots\end{gathered}$$

we finally obtain the inequality

$$\delta c^{(n)} \leqslant \delta c^{(n-1)}(B + b_1(c^{(n-1)} + c^{(n-2)}))/\Delta, \ c^{(1)} \leqslant b_2$$

Hence follows the majorizing scalar equation /3/

$$p = (b_2 + b_1 p + Bp^2)/\Delta$$

and a sufficient condition of convergence of the process (2.2)

$$\Delta \geqslant B + 2\sqrt{b_1 b_2} \tag{2.3}$$

The first non-trivial correction to the characteristic exponent is

$$\Delta\lambda^{(2)} = \Sigma b_{1l}{}^T U_{-l} b_{2-l}$$

3. *Iterative solution of the reducibility problem.* The solution of the problem /4, 5/ of the reducibility of Eq.(1.2) is equivalent to determining the constant matrix C which is kinematically similar to $\Lambda_0 + B(t)$. First consider the case when the matrix C is diagonalizable. We need to find a periodic transformation matrix $R(t)$ that satisfies the equation

$$(\Lambda_0 + B(t))R(t) - R^{\cdot}(t) = R(t)\Lambda, \ \Lambda = \operatorname{diag}\{\lambda_j\} \tag{3.1}$$

The solution of Eq.(3.1) simultaneously produces the characteristic exponents of Eq.(1.2). As before, the matrix $B(t)$ is assumed to be small, and therefore $R(t)$ is close to the identity matrix: $R(t) = E + X(t)$.

Eq.(3.1) can be solved by a Newtonian iteration algorithm. To this end, consider the relationship

$$\begin{gathered}(\Lambda^{(k-1)} + B^{(k-1)})(E + X^{(k)}) - X^{(k)\cdot} = (E + X^{(k)})(\Lambda^{(k)} + B^{(k)}) \\ k = 1, 2, \ldots, \ \Lambda^{(0)} = \Lambda_0, \ B^{(0)} = B(t)\end{gathered} \tag{3.2}$$

We assume that

$$B^{(k)} = \Sigma B_l^{(k)} e^{ilt}, \quad X^{(k)} = \Sigma X_l^{(k)} e^{ilt}, \quad \Lambda^{(k)} = \mathrm{diag}\,\{\lambda_j^{(k)}\} = \mathrm{const}$$

The matrix $B^{(k)}$ in (3.2) is chosen from the condition of a quadratic rate of convergence of the iterations. As a result, we obtain equations for $X^{(k)}$, $B^{(k)}$ and $\Lambda^{(k)}$:

$$X^{(k)'} + X^{(k)}\Lambda^{(k-1)} - \Lambda^{(k-1)}X^{(k)} = B^{(k-1)} - \Delta\Lambda^{(k)} \tag{3.3}$$
$$B^{(k)} = (E + X^{(k)})^{-1}(B^{(k-1)}X^{(k)} - X^{(k)}\Delta\Lambda^{(k)}), \quad \Lambda^{(k)} = \Lambda^{(k-1)} + \Delta\Lambda^{(k)}$$

The matrix $\Delta\Lambda^{(k)}$ is chosen from the conditions for a periodic solution for the first equation of system (3.3) to exist. A necessary condition for this is that the Fourier expansions of the diagonal elements of the matrix $B^{(k-1)} - \Delta\Lambda^{(k)}$ do not contain zeroth terms, and if all the diagonal elements of the matrix $\Lambda^{(k-1)}$ satisfy the inequalities

$$\lambda_n^{(k-1)} - \lambda_j^{(k-1)} \neq il, \ l = 0, \pm 1, \pm 2, \ldots, \ n \neq j$$

then this condition is also sufficient. In this case, $\Delta\Lambda^{(k)} = \mathrm{diag}\,\{b_{0jj}^{(k-1)}\}$ where $b_{0jj}^{(k-1)}$ are diagonal elements of the matrix $B_0^{(k-1)}$.

The elements $x_{nj}^{(k)}$ of the matrix $X^{(k)}$ satisfy the equations

$$x_{nj}^{(k)'} + \Delta\lambda_{nj}^{(k-1)} x_{nj}^{(k)} = \Sigma b_{lnj}^{(k-1)} e^{ilt}, \quad n \neq j$$
$$x_{jj}^{(k)'} = \Sigma b_{ljj}^{(k-1)} e^{ilt}, \quad \Delta\lambda_{nj}^{(k-1)} = \lambda_j^{(k-1)} - \lambda_n^{(k-1)}$$

whose solution under the above conditions have the form

$$x_{nj}^{(k)} = \sum_l \frac{b_{lnj}^{(k-1)}}{\Delta\lambda_{nj}^{(k-1)} + il} e^{ilt}, \quad x_{jj}^{(k)} = \sum_l \frac{b_{ljj}^{(k-1)}}{il} e^{ilt}$$

If the iterations (3.3) converge, they converges almost everywhere at a quadratic rate. $B^{(k)}$ and $X^{(k)}$ tend to zero, the diagonal elements of $\Lambda^{(k)}$ tend to the characteristic exponents, and the product

$$R_0 (E + X^{(1)})(E + X^{(2)})(E + X^{(3)}) \ldots$$

tends to the matrix $R(t)$. As in Sect.2, convergence of the iterations (3.3) imposes a lower limit of the form (2.3) on the minimum distance between the characteristic exponents. The sufficient condition for the iterations to converge has the form

$$|\lambda_{0n} - \lambda_{0j} - il| \geqslant 4.56 B_s \ l = 0, \pm 1, \pm 2, \ldots, \ n \pm j \tag{3.4}$$

Here B is the norm of the matrix $B(t)$ introduced in Sect.2.

Note that (3.3) requires inverting the matrix $E + X^{(k)}$, which is time-dependent and is defined by a Fourier series (in practice, the series contains a finite number of terms). The inverse should be computed either in the form of a series

$$(E + X^{(k)})^{-1} = E - X^{(k)} + X^{(k)2} - X^{(k)3} + \ldots$$

that converges at the rate of a geometrical progression for $\|X^{(k)}\| < 1$ or in the form of a product

$$(E + X^{(k)})^{-1} = (E - X^{(k)})(E + X^{(k)2})(E + X^{(k)4}) \ldots$$

that converges at a quadratic rate under the same conditions.

4. The existence of close multipliers. The case when the transformation matrix becomes singular during the iteration process. Let us modify algorithm (3.3) so as to allow for the existence of close generating eigenvalues. In this case, the matrix $E + X^{(k)}$ may become singular during the iteration process. The problem of diagonalization should be replaced with the problem of reduction to a block-diagonal constant matrix in canonical form with either quasi-Jordan or triangular blocks along the diagonal. The generating matrix Λ_0 is also reduced to canonical form.

The modified algorithm has the form

$$X^{(k)'} + X^{(k)}\Lambda^{(k-1)} - \Lambda^{(k-1)}X^{(k)} = B^{(k-1)} - \Delta\Lambda_1^{(k)} \tag{4.1}$$
$$B_1^{(k)} = (E + X^{(k)})^{-1}(B^{(k-1)}X^{(k)} - X^{(k)}\Delta\Lambda_1^{(k)})$$
$$\Lambda_1^{(k)} = \Lambda^{(k-1)} + \Delta\Lambda_1^{(k)}, \quad B^{(k)} = R^{(k)-1} B_1^{(k)} R^{(k)}$$
$$\Lambda^{(k)} = R^{(k)-1}(\Lambda_1^{(k)} R^{(k)} - R^{(k)'}); \quad k = 1, 2, \ldots, \quad \Lambda^{(0)} = \Lambda_0, \quad B^{(0)} = B(t)$$

The matrix $\Delta\Lambda_1^{(k)}$ is chosen not only from the conditions for a bounded solution of the first equation to exist, but also so as to eliminate small denominators in the solution. As we will show, it can be chosen as a block-diagonal constant matrix. In the next iteration step $\Lambda^{(k-1)} + \Delta\Lambda_1^{(k)}$ is reduced to the required canonical form.

Let us demonstrate the application of algorithm (4.1) by considering an example. Assume that the convergence condition (3.4) is violated for three generating eigenvalues $\lambda_{10}, \lambda_{20}$ and λ_{30}, i.e., integers p and q exist such that

$$|\lambda_{20} - \lambda_{10} + ip| < 4.56\,B,\ |\lambda_{30} - \lambda_{20} + iq| < 4.56B,$$
$$|\lambda_{30} - \lambda_{10} + i(p+q)| < 4.56\,B$$

and in all other cases (3.4) is satisfied.

To avoid the appearance of small denominators, we should include in the matrix $\Delta\Lambda_1^{(k)}$ a block of the form

$$\begin{Vmatrix} b_{011}^{(k-1)} & b_{p12}^{(k-1)} e^{ipt} & b_{p+q13}^{(k-1)} e^{i(p+q)t} \\ b_{-p21}^{(k-1)} e^{-ipt} & b_{022}^{(k-1)} & b_{q23}^{(k-1)} e^{iqt} \\ b_{-p-q31}^{(k-1)} e^{-i(p+q)t} & b_{-q32}^{(k-1)} e^{-iqt} & b_{033}^{(k-1)} \end{Vmatrix}$$

We then determine the matrix $R^{(k)}$ which transforms $\Lambda_1^{(k)}$ to a kinematically similar constant block-diagonal matrix $\Lambda^{(k)}$ of the above form. The transformation is carried out in two stages, i.e., $R^{(k)}$ is represented as the product of two matrices: $R^{(k)} = R_1^{(k)} R_2^{(k)}$. In the first stage, using the matrix

$$R_1^{(k)} = \operatorname{diag}\{1, e^{-ipt}, e^{-i(p+q)t}, 1, \ldots, 1\} \tag{4.2}$$

we apply a similarity transformation transforming $\Lambda_1^{(k)}$ to a constant matrix $\Lambda_2^{(k)}$ with the diganonal block

$$\begin{Vmatrix} \lambda_1^{(k-1)} + b_{011}^{(k-1)} & b_{p12}^{(k-1)} & b_{p+q13}^{(k-1)} \\ b_{-p21}^{(k-1)} & \lambda_2^{(k-1)} + ip + b_{022}^{(k-1)} & b_{q23}^{(k-1)} \\ b_{-p-q31}^{(k-1)} & b_{-q32}^{(k-1)} & \lambda_3^{(k-1)} + i(p+q) + b_{033}^{(k-1)} \end{Vmatrix}$$

and $R_2^{(k)}$ transforms $\Lambda_2^{(k)}$ to the form $\Lambda^{(k)}$, i.e., the problem has been actually reduced to examining a constant 3x3 matrix.

In general, this method can be used if a so-called condition of separability of the characteristic exponents $\lambda_{0j} + il$ $(l = 0, \pm 1, \pm 2, \ldots)$ corresponding to the generating eigenvalues is satisfied.

Consider a group of characteristic exponents that are close (in the sense of violation of the convergence condition). The group is constructed so that for each element there exists at least one close element in the group and that condition (3.4) is satisfied for any two characteristic exponents from different groups. We say that the separability condition is satisfied if the groups of close characteristic exponents constructed in this way are covered in the complex plane by non-intersecting open simply connected bounded sets. In this case, two characteristic exponents corresponding to the same generating eigenvalue and having different imaginary parts cannot belong to the same group. The separability condition imposes a relatively weak upper limit on the norm B of the perturbing matrix $B(t)$: this upper limit is always weaker than the inequality $4.56B < 1/m$, where m is the dimension of the matrix of the given problem.

Suppose that the separability condition is satisfied. Only characteristic exponents of the same group may merge during the iteration process (4.1); the difficulty with small denominators is also localized within group. Partition the groups of close characteristic exponents into non-intersecting classes, such that the groups in each class can be obtained as a result of a translation by an integer number of units along the complex axis. The classes constructed in this way are characterized by disjoint sets of generating eigenvalues. Within each set, it is generally the multipliers $\exp(2\pi\lambda_{0j})$, and not the corresponding eigenvalues, that are close. Among all groups in a special class, designate one main group. This may be the group in which the maximum number of elements is equal to the eigenvalue λ_{0j}.

For each class, form a block in the matrix Λ_0 whose diagonal elements are the generating eigenvalues from the set corresponding to this class. If close eigenvalues exist, the blocks in general are triangular, and not diagonal. Then it is helpful to apply a kinematic similarity transformation to the matrix $\Lambda_0 + B(t)$ in order to replace the generating eigenvalues in the blocks by the characteristic exponents from the main groups. The transformation matrix is diagonal and its form is like (4.2). This ends the preliminary analysis and the preliminary transformations, and we can proceed with the iterations (4.1).

Let us demonstrate the construction of the block-diagonal matrix $\Delta\Lambda_1^{(k)}$. To the isolated characteristic exponent $\lambda_i^{(k-1)}$ not included in any of the groups there corresponds a unique diagonal element of $\Delta\Lambda_1^{(k)}$, equal to the diagonal element of the matrix $B_0^{(k-1)}$ - the zeroth term in the Fourier series expansion of $B^{(k-1)}(t)$. For each main group of characterisitics exponents, there is a diagonal block in $\Delta\Lambda_1^{(k)}$, which is equal to the corresponding diagonal block of $B_0^{(k-1)}$. The matrix is thus constant and the kinematic similarity transformation suggested in (4.1) may be replaced by $R^{(k)-1}\Lambda_1^{(k)}R^{(k)}$ with the constant transformation matrix $R^{(k)}$. The matrix $\Lambda_1^{(k)}$ is thus block-diagonal, and we have to transform its separate blocks to canonical form, which in many practical cases are quite small. This can be done, for instance, by the QR-algorithm with translation or, if the blocks are small, by perturbation theory.

If the parametric excitation matrix $B(t)$ is not small and the separability condition is not satisfied, then we consider the matrix $\Lambda_0 + \gamma B(t)$ that depends on the parameter $\gamma \in [0, 1]$. Applying the iterations (4.1) to this matrix with stepwise increments of γ, we obtain the final result for $\gamma = 1$. The parameter increment is chosen to be sufficiently small, so that the separability condition is satisfied.

5. *Example.* Consider an electromagnet moving with constant velocity v_0 along a guideway formed by dynamically uncoupled elastic homogeneous beams of mass m_1 and length l each (Fig.1). The problem corresponds to the simplest model of a magnetically suspended vehicle /1, 2, 6/*. (*See also POGORELOV D.YU., Spectral Analysis of Weakly Perturbed Linear Dynamic Systems, Bryansk, 1988. Unpublished manuscript, VINITI 03.10.88, 7261-1388.)

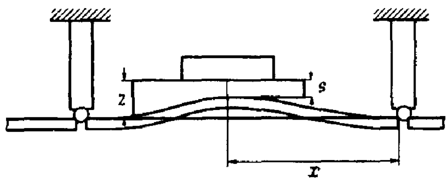

Fig.1

The stability of the suspension is ensured by regulating the clearance s between the pole terminals of the electromagnetic and the beam. The control uses the readings of the clearance sensor $s - s_0$ and the accelerometer $z^{\cdot\cdot}$ (s_0 is the nominal clearance). The clearance and the vertical coordinate z are related by the equation $s = z + w(x, t) + \xi(x)$, where w is the beam deflection and $\xi(x)$ is the function describing the unevenness of the guideway.

Ignoring eddy currents and saturation, we obtain from the Lagrange-Maxwell equations the equations of the electromagnet

$$mz^{\cdot\cdot} = {}^1/_2\,(dL/ds)j^2 + Mg,\quad Lj^{\cdot} + (dL/ds)js^{\cdot} = -rj + u$$
$$u = U_0 + a_0(s - s_0) + a_1 s^{\cdot} + a_2 z^{\cdot\cdot},\quad L = L_p s_0/s + L_s$$

Here j, u, r, L are the current, the output voltage, the ohmic resistance, and the inductance of the electromagnetic winding, M is the vehicle mass and a_0, a_1 and a_2 are the regulator parameters. The equations of the model are closed by the equation of bending vibrations of the beam. Assuming that the beam length is large compared with the characteristic dimension of the cross-section, we will use the equations of the technical theory of bending vibrations of rods.

Let us linearize the equations relative to the stationary values of the variables z_0, s_0, J_0, U_0 and $w_0(x)$ and change to dimensionless variables and parameters. We also expand the dimensionless variation of the deflection $\Delta w = (w - w_0)/s_0$ in orthonormal eigenforms $\varphi_k(x')$ of the bending vibrations of the beam (ignoring the interaction with the magnet)

$$\Delta w = \sum_{k=1}^{\infty} w_k \varphi_k(x'),\quad x' = \frac{x}{l}$$

The equations of bending vibrations of the beam may be replaced by an infinite-dimensional system of second-order ordinary differential equations /7/. As a result, we obtain the following system of variational equations:

$$\Delta z^{\cdot\cdot} = c\Delta s - \Delta j \tag{5.1}$$
$$\Delta j^{\cdot} - (1 - d)\Delta s^{\cdot} + \Delta j/T_1 = c_0\Delta s + c_1\Delta s^{\cdot} + c_2\Delta z^{\cdot\cdot}$$
$$w_k^{\cdot\cdot} + \omega_k^2 w_k = -\mu\varphi_k(x'(\tau))\Delta z^{\cdot\cdot},\quad k = 1, 2, \ldots$$

$$\Delta z = \Delta s + \xi(x'(\tau)) + \sum_{k=1}^{\infty} w_k(\tau)\varphi_k(x'(\tau)), \quad x'(\tau) = \frac{v_0 T_0 \tau}{l}$$

Here

$$\Delta z = \frac{z - z_0 - w_0(x'(\tau))}{s_0}, \quad \Delta s = \frac{s - s_0}{s_0}, \quad \Delta j = \frac{j - J_0}{J_0}, \quad T_0^2 = \frac{ms_0}{2Mg}$$

$$T_1 = \frac{L_s + L_p}{rT_0}, \quad d = \frac{L_s}{L_p}, \ \mu = \frac{m}{m_1}, \quad c_0 = \frac{a_0 s_0}{U_0 T_1}, \quad c_1 = \frac{a_1 s_0}{U_0 T_1 T_0},$$

$$c_2 = \frac{a_2 s_0}{U_0 T_1 T_0^2}$$

The dimensionless time $\tau = t/T_0$ has been introduced. In (5.1) and below, the dot denotes differentation with respect to τ and ω_k are the eigenfrequencies of the beam corresponding to the forms $\varphi_k(x')$.

At instants of time $\tau = nl/(v_0 T_0)$ $(n = 1, 2, \ldots)$, the variables w_k have a discontinuity because the electromagnet reaches a stationary guideway beam:

$$w_k(nT + 0) = 0, \ T = l/(v_0 T_0), \ n = 1, 2, \ldots \tag{5.2}$$

The guideway unevenness is modelled by a random function generated by a shaping filter:

$$T\xi^{\cdot} + \alpha\xi = \varkappa V(\tau). \tag{5.3}$$

Here α and $\varkappa$ are the filter parameters and $V(\tau)$ is white noise of unit intensity.

If we consider only a finite number of eigenforms of beam vibrations, Eqs.(5.1)-(5.3) may be written in the form

$$x^{\cdot} = A(\tau)x + bV(\tau), \ A(\tau + T) = A(\tau) \tag{5.4}$$

$$x(nT + 0) = Gx(nT - 0), \quad G = \begin{vmatrix} E & 0 \\ 0 & 0 \end{vmatrix}$$

$$A(\tau) = \begin{vmatrix} A_{11} & A_{12}(\tau) \\ A_{21}(\tau) & A_{22} \end{vmatrix} \begin{matrix} A_{12}(\tau) = ea_{12}^T(\tau) \\ A_{21}(\tau) = a_{21}(\tau)e^T, \quad e = (0, 0, 1, 0)^T \end{matrix}$$

$$x = (z, z^{\cdot}, z^{\cdot\cdot}, \xi, w_1, w_1^{\cdot}, w_2, w_2^{\cdot}, \ldots)^T$$

$$b = (0, 0, b_1\varkappa/T, \varkappa/T, 0, \ldots, 0)^T$$

$$A_{11} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -b_0 & -b_1 & -b_2 & b_1\alpha/T + b_0 \\ 0 & 0 & 0 & -\alpha/T \end{vmatrix} \begin{matrix} A_{22} = \operatorname{diag}\{C_j\} \\ C_j = \begin{vmatrix} 0 & 1 \\ -\omega_j^2 & 0 \end{vmatrix} \end{matrix}$$

$$a_{12}(\tau) = (b_0\varphi_1(x'(\tau)), \ b_1\varphi_1(x'(\tau)), \ b_0\varphi_2(x'(\tau)), \ b_1\varphi_2(x'(\tau)), \ldots)^T$$

$$a_{21}(\tau) = -\mu(0, \varphi_1(x'(\tau)), 0, \varphi_2(x'(\tau)), \ldots)^T$$

$$b_0 = c_0 - c/T_1, \ b_1 = c_1 + 1 - c - d, \ b_2 = c_2 + 1/T_1$$

As the unperturbed matrix, we consider the block-diagonal matrix $A_0 = \operatorname{diag}\{A_{11}, A_{22}\}$. The generating eigenvalues and eigenvectors are determined by the blocks A_{11} and A_{22}.

Let us carry out a covariance analysis of problem (5.4). Making the change of variables $x = R(\tau)y$, we rewrite Eq.(5.4) in the form

$$\begin{gathered} y^{\cdot} = \Lambda y + c(\tau)V, \ y(nT + 0) = Hy(nT - 0) \\ c(\tau) = R^{-1}(\tau)b, \ H = R^{-1}(0)GR(0) \end{gathered} \tag{5.5}$$

Eq.(5.5) has the matricant $u(\tau) = \exp(\Lambda\tau)$ and its general solution, allowing for discontinuity of the variables w_k, is given by /6/

$$y(nT + \tau) = u(\tau)K^n y_0 + u(\tau)\left\{\sum_{m=1}^{n} K^m \int_0^T u(-t)c(t)V(t + (n - m)T)\,dt\right\} +$$

$$\int_0^\tau u(\tau - t)c(t)V(t + nT)\,dt, \quad K = Hu(T), \tau \in [0, T)$$

Let us now compute the covariance matrix $P(\tau)$:

$$P(nT + \tau) = u(\tau)\{K^n P_0 K^{*n} + \sum_{m=1}^{n} K^m D(T) K^{*m} + D(\tau)\} u^*(\tau)$$

$$D(\tau)=\int_0^\tau u(-t)\,c(t)\,c^*(t)\,u^*(-t)\,dt$$

(the asterisk denotes conjugation of matrices). The steady-state value is obtained by passing to the limit $n\to\infty$:

$$P_s(nT+\tau)=u(\tau)\{\sum_{m=1}^{\infty}K^m D(T)K^{*m}+D(\tau)\}u^*(\tau) \tag{5.6}$$

We will only consider the case when the matricant $u(\tau)$ is diagonal. Then the integral $D(\tau)$ is evaluated analytically.

Consider the matrix $Q=\Sigma K^m D(T)K^{*m}$, whose elements are in general complex. We change to real matrices by the transformation

$$Q'=R(0)QR^*(0)=\Sigma K'^m D'(T)K'^{Tm}$$

$$K'=R(0)KR^{-1}(0)=Gu'(T)=\begin{vmatrix}u_{11}' & u_{12}'\\ 0 & 0\end{vmatrix}$$

$$D'(T)=R(0)D(T)R^*(0)$$

Here $u'(\tau)$ is the matricant of the original system (5.4). Q' is computed by reducing K' to the diagonal form $\Lambda'=S^{-1}K'S$, such that the diagonal elements of Λ' are the multipliers of problem (5.4) (some of the multipliers are zero while others are assumed to be different). The matrix Q' takes the form

$$Q'=S\{\Sigma\Lambda'^m S^{-1}D'(T)S^{*-1}\Lambda'^{*m}\}S^*$$

and the sum in braces is easily calculated. Q is determined from Q' and $P_s(\tau)$ is then obtained from (5.6).

The steady-state value $P_s'(\tau)=\langle xx^T\rangle$ of the covariance matrix of the original problem is related to $P(\tau)$ by the equality

$$P_s'(\tau)=R(\tau)P_s(\tau)R^*(\tau)$$

The results of numerical calculations are presented in Figs.2-5. We used the following values of the parameters in these calculations (all are given in dimensionless form): $d=0.5$; $c=0.7$; $\alpha=2$; $\varkappa=1$; $\omega_1=2.2$; $c_0=0.7$; $c_1=2$; $c_2=1.5$; $\mu=1$; $T_1=7$. The guideway was modelled by beams with hinged support at the ends. The matrix Fourier series were limited to 19 terms.

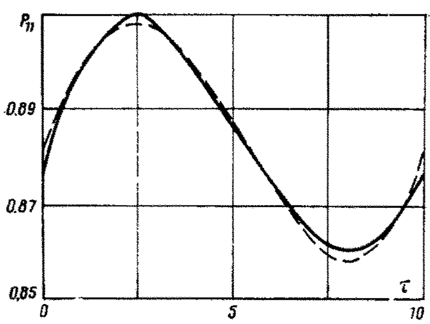

Fig.2

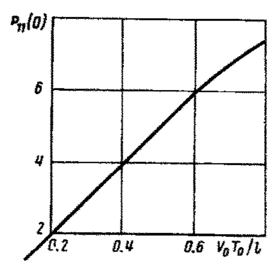

Fig.3

Fig.2 shows the steady-state variance $P_{11}(\tau)$ of the vertical coordinate Δz as a function of time for the dimensionless period $T=10$. The variance does not change much over time. The dependence of $P_{11}(0)$ on the velocity v_0 is shown in Fig.3. The following velocity range was assumed: $v_0T_0/l\in[0,1;0,8]$. For smaller velocities, the effect of longitudinal motion is weak, and therefore a sufficiently good approximation is obtained by analysing the dynamics of a vehicle which is stationary in the longitudinal direction.

The dependence of the characteristic exponents of system (5.4) on the velocity is shown in Fig.4. Of the complete series of exponents $\lambda_j+2\pi ik/T$, we show only one exponent with the maximum norm $\|r_{jk}\|$ of the corresponding vector. At high velocities, the characteristic exponents may have positive real parts, but this does not cause instability of the system. The vehicle periodically reaches another stationary element of the guideway and the system multipliers, whose dependence on the velocity is shown in Fig.5 (the dark circles), satisfy the stability conditions; the light circles show for comparison the values of $\exp(\lambda_j T)$, where λ_j is the characteristic exponent.

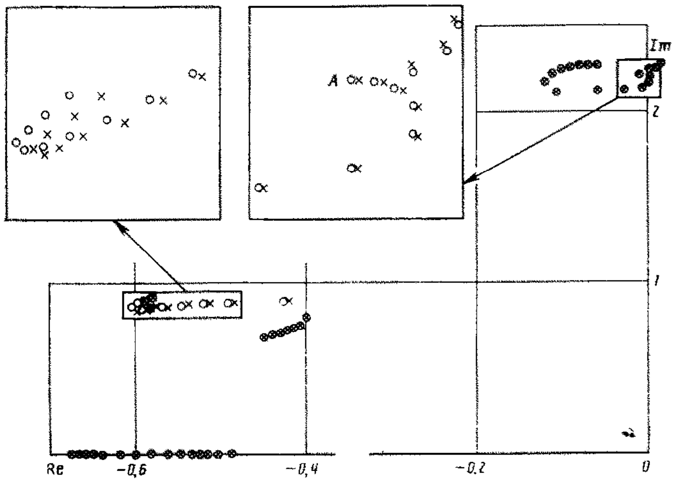

Fig.4

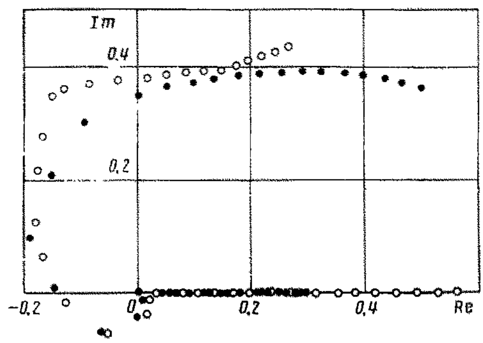

Fig.5

Note that the curves of the characteristic exponents as a function of velocity (Fig.4) are not smooth. The point A corresponds to the merging of characteristic exponents, which causes singularity of the matrix $E + X^{(k)}$ during the iterations (3.3) and requires the use of the generalized algorithm (4.1). We analysed the effect of the number of eigenforms of beam vibrations included in (5.4) on the accuracy of the results. The solid curve in Fig.2 corresponds to two forms and the broken curve to one form. As in Fig.4, the crosses correspond to two eigenforms and the light circles to one form. Thus, allowance for one eigenform of beam vibrations produces qualitatively and quantitatively reliable estimates of the characteristics. For this reason, in particular, the use of Timoshenko's improved theory of bending vibrations of rods does not change the result for our thin beams appreciably.

Note that the algorithm of Sects.3 and 4 has a number of advantages compared to other numerical methods /8/. For instance, its convergence is faster. Compared with the method of monodromy matrices, it does not require numerical integration of a system of equations and in addition it solves the reducibility problem. Hill's method of determinants involves large auxiliary matrices, which in turn increase the computing time.

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THE CONSTRUCTION OF SUCCESSIVE APPROXIMATIONS OF THE PERTURBATION METHOD FOR SYSTEMS WITH RANDOM COEFFICIENTS*

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A successive approximation procedure is proposed for stochastic systems reducible to standard form with non-"white noise" perturbations. To a first approximation, the solution of the perturbed system converges to the solution of some averaged deterministic system, and to a second approximation it converges to the solution of some averaged diffusion equation. Higher approximations enable one to estimate the deviations from a diffusion process. The convergence interval depends on the properties of the deterministic solution of the first-approximation equation.

1. We consider systems with equations of motion reducible to the standard form

$$x^{\cdot} = \varepsilon F(t, x) + \varepsilon^2 G(t, x),\ x(0) = a \in R_n \tag{1.1}$$

Here ε is a small parameter. For a fixed x, the functions $F(t, \cdot)$ and $G(t, \cdot)$ are stochastic processes with expectations $MF(t, \cdot) = f(t, \cdot)$, $MG(t, \cdot) = g(t, \cdot)$.

Henceforth, we assume that the functions f, g are periodic or conditionally periodic in t and the means

$$\bar{F}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x)\, dt \tag{1.2}$$

exist uniformly in $x \in S \subset R_n$; the function $\bar{G}(x)$ is defined similarly. Other restrictions on the coefficients of system (1.1) are stated below.

So far, two special cases of system (1.1) have been considered /1, 2/.

a) $\bar{F}(x) \not\equiv 0$. Then /1/ under appropriate restrictions the solution $x(t, \varepsilon) = x_\varepsilon(\tau_1)$ of system (1.1) weakly converges /3/ as $\varepsilon \to 0$ to a deterministic process $x_0(\tau_1)$ — the solution of the equation

$$dx_0/d\tau_1 = \bar{F}(x_0),\ x_0(0) = a,\ \tau_1 = \varepsilon t \tag{1.3}$$

If the solution $x_0(\tau_1)$ of Eq.(1.3) is asymptotically stable, then the convergence $x_\varepsilon \to x_0$ is ensured for $0 \leqslant \tau_1 < \infty$ /4/. If stability is not required, $x_\varepsilon \to x_0$ for $0 \leqslant \tau_1 \leqslant T_1$, where

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